Applied Mathematics and Mechanics (English Edition), 2006, **27**(7):871–882
©Editorial Committee of Appl. Math. Mech., **ISSN** 0253-4827

INVESTIGATION OF BEHAVIOR OF MODE-I INTERFACE CRACK IN PIEZOELECTRIC MATERIALS BY USING SCHMIDT METHOD *

ZHOU Zhen-gong (周振功)[1], WANG Biao (王彪)[2]

(1. Center for composite materials, Harbin Institute of Technology, Harbin 150001, P. R. China;
2. School of Physics and Engineering, Sun Yat-Sen University, Guangzhou 510275, P. R. China)

(Contributed by WANG Biao)

Abstract: The behavior of a Mode-I interface crack in piezoelectric materials was investigated under the assumptions that the effect of the crack surface overlapping very near the crack tips was negligible. By use of the Fourier transform, the problem can be solved with the help of two pairs of dual integral equations. To solve the dual integral equations, the jumps of the displacements across the crack surfaces were expanded in a series of Jacobi polynomials. It is found that the stress and the electric displacement singularities of the present interface crack solution are the same as ones of the ordinary crack in homogenous materials. The solution of the present paper can be returned to the exact solution when the upper half plane material is the same as the lower half plane material.

Key words: interface crack; intensity factors; piezoelectric materials

Chinese Library Classification: O346.53
2000 Mathematics Subject Classification: 74J20; 74R10; 74F17
Digital Object Identifier(DOI): 10.1007/s 10483-006-0702-y

Introduction

It is very important to investigate the behavior of an interface crack in piezoelectric materials, because the interface crack is the main reason leading to the constructive failure for various devices related with the materials. An interface crack between isotropic non-piezoelectric materials had been actively studied for many years. Particularly, an oscillating interface crack model has been developed in Ref.[1], and re-examined in Ref.[2]. To eliminate the physically unrealistic oscillating singularity at the crack tips, the contact interface crack model has been suggested in Ref.[3], and was also developed later in numerical and analytical manner in Refs.[4,5]. For the interface crack in the piezoelectric materials, the electrically conducting and electrically insulated crack faces were considered in Ref.[6], the singularities at the tips of an interface crack were investigated. Particularly, a new type of singularity of a real power type was discovered around an interface crack tip for insulated crack faces. A permeable interface crack with an artificial contact zone at the right-hand side crack tip between two piezoelectric semi-infinite half-planes is considered by use of the Stroh method under remote mixed-mode loading in Ref.[7]. However, it is found that these solutions contain the stress and electric displacement oscillatory singularity near the crack tips in Refs.[1,2,6,7], except for a certain class of composites. This is quite different from ordinary crack in homogeneous materials and not reasonable

* Received Jan.6, 2005; Revised Mar.21, 2006
Project supported by the National Natural Science Foundation of China (Nos.10572155, 10572043); the Natural Science Foundation for Distinguished Young Scholars of Heilongjiang Province of China (No.JC04-08)
Corresponding author ZHOU Zhen-gong, Professor, Doctor, E-mail: zhouzhg@hit.edu.cn

according to the physical nature. Therefore, in comparison with the ordinary crack problems, it is difficult to analysis accurately the interface crack problems and there are not enough the date of stress intensity factors for interface cracks. This is a disputed question for a long time. Up to the present, the problem has not been solved completely.

Mathematically, the solutions in Refs.[1,2] are exact forms in spite of the incomprehensibility in fracture mechanics. However, from an engineering viewpoint, it is more desirable to seek a solution that is physically acceptable. In the present paper, the similar problem which was treated by Herrmann[7] was reworked by use of a somewhat different method, named as Schmidt method[8]. As in the previous study[9], in this paper, it was also assumed that the effect of the crack surface overlapping very near the crack tips is negligible. By use of the Fourier transform, the problem can be solved with the help of two pairs of dual integral equations in which the unknown variables are the jumps of the displacements across the crack surfaces. To solve the dual integral equations, the jumps of the displacements across crack surfaces were expanded in a series of Jacobi polynomials. This process was quite different from those adopted in Refs.[1–7] mentioned above. However, in the previous works[1−7], the unknown variables of dual integral equations were the dislocation density functions. This is the major difference. Numerical examples are provided to show the effects of the length of the crack, the material properties upon the stress and the electric displacement intensity factors of the cracks. Contrary to the previous solution of the interface crack, it is found that the stress singularity of the present interface crack solution is of the same nature as that for the ordinary crack in homogeneous materials. The exact solution can be obtained when the upper half plane material is the same as the lower half plane material.

1 Formulation of Problem

It is assumed that there is an interface crack of length $2l$ along the x-axis between two dissimilar piezoelectric materials half-planes as shown in Fig.1.

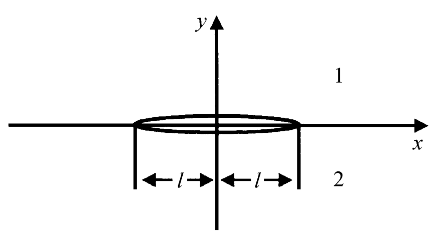

Fig.1 Geometry and coordinate system for the interface crack

The constitutive relations for piezoelectric materials polarized along the y-direction exhibiting transversely isotropic behavior (hexagonal symmetry) can be written as

$$\sigma_x^{(j)}(x,y) = c_{11}^{(j)}\frac{\partial u^{(j)}}{\partial x} + c_{13}^{(j)}\frac{\partial v^{(j)}}{\partial y} + e_{31}^{(j)}\frac{\partial \phi^{(j)}}{\partial y}, \tag{1}$$

$$\sigma_y^{(j)}(x,y) = c_{13}^{(j)}\frac{\partial u^{(j)}}{\partial x} + c_{33}^{(j)}\frac{\partial v^{(j)}}{\partial y} + e_{33}^{(j)}\frac{\partial \phi^{(j)}}{\partial y}, \tag{2}$$

$$\sigma_{xy}^{(j)}(x,y) = c_{44}^{(j)}(\frac{\partial u^{(j)}}{\partial y} + \frac{\partial v^{(j)}}{\partial x}) + e_{15}^{(j)}\frac{\partial \phi^{(j)}}{\partial x}, \tag{3}$$

$$D_x^{(j)}(x,y) = e_{15}^{(j)}(\frac{\partial u^{(j)}}{\partial y} + \frac{\partial v^{(j)}}{\partial x}) - \varepsilon_{11}^{(j)}\frac{\partial \phi^{(j)}}{\partial x}, \tag{4}$$

$$D_y^{(j)}(x,y)=e_{31}^{(j)}\frac{\partial u^{(j)}}{\partial x}+e_{33}^{(j)}\frac{\partial v^{(j)}}{\partial y}-\varepsilon_{33}^{(j)}\frac{\partial \phi^{(j)}}{\partial y}, \tag{5}$$

where $(\sigma_x^{(j)},\sigma_y^{(j)},\sigma_{xy}^{(j)})$ and $(D_x^{(j)},D_y^{(j)})$ are the components of stress tensor and electric displacement vector, $(u^{(j)},v^{(j)})$ and $\phi^{(j)}$ are the components of the displacement vector and electric potential, $c_{11}^{(j)},c_{13}^{(j)},c_{33}^{(j)},c_{44}^{(j)}$ are the elastic stiffness constants measured in a constant electric field, $\varepsilon_{11}^{(j)},\varepsilon_{33}^{(j)}$ are the dielectric constants measured at constant strain, $e_{15}^{(j)},e_{31}^{(j)},e_{33}^{(j)}$ are the piezoelectric constants, and the superscript $j=1,2$ correspond to the half-planes $y\geq 0$ and $y\leq 0$ through in this paper. For two-dimensional piezoelectric coupling problems in plane strain, the governing equations are obtained as

$$c_{11}^{(j)}\frac{\partial^2 u^{(j)}}{\partial x^2}+c_{44}^{(j)}\frac{\partial^2 u^{(j)}}{\partial y^2}+(c_{13}^{(j)}+c_{44}^{(j)})\frac{\partial^2 v^{(j)}}{\partial x\partial y}+(e_{31}^{(j)}+e_{15}^{(j)})\frac{\partial^2 \phi^{(j)}}{\partial x\partial y}=0, \tag{6}$$

$$c_{44}^{(j)}\frac{\partial^2 v^{(j)}}{\partial x^2}+c_{33}^{(j)}\frac{\partial^2 v^{(j)}}{\partial y^2}+(c_{13}^{(j)}+c_{44}^{(j)})\frac{\partial^2 u^{(j)}}{\partial x\partial y}+e_{15}^{(j)}\frac{\partial^2 \phi^{(j)}}{\partial x^2}+e_{33}^{(j)}\frac{\partial^2 \phi^{(j)}}{\partial y^2}=0, \tag{7}$$

$$(e_{31}^{(j)}+e_{15}^{(j)})\frac{\partial^2 u^{(j)}}{\partial x\partial y}+e_{15}^{(j)}\frac{\partial^2 v^{(j)}}{\partial x^2}+e_{33}^{(j)}\frac{\partial^2 v^{(j)}}{\partial y^2}-\varepsilon_{11}^{(j)}\frac{\partial^2 \phi^{(j)}}{\partial x^2}-\varepsilon_{33}^{(j)}\frac{\partial^2 \phi^{(j)}}{\partial y^2}=0. \tag{8}$$

For the interface crack, the boundary conditions can be written as

$$\sigma_{xy}^{(1)}(x,0^+)=\sigma_{xy}^{(2)}(x,0^-)=0,\quad \sigma_y^{(1)}(x,0^+)=\sigma_y^{(2)}(x,0^-)=-\sigma_0,\quad |x|\leq l, \tag{9}$$

$$\begin{cases} u^{(1)}(x,0^+)=u^{(2)}(x,0^-),\quad v^{(1)}(x,0^+)=v^{(2)}(x,0^-),\\ \sigma_y^{(1)}(x,0^+)=\sigma_y^{(2)}(x,0^-),\quad \sigma_{xy}^{(1)}(x,0^+)=\sigma_{xy}^{(2)}(x,0^-),\end{cases}\quad |x|>l, \tag{10}$$

$$\phi^{(1)}(x,0^+)=\phi^{(2)}(x,0^-),\quad D_y^{(1)}(x,0^+)=D_y^{(2)}(x,0^-),\quad |x|\geq 0, \tag{11}$$

where σ_0 is a magnitude of the uniform stress loading.

2 Solution Procedure

Because of the symmetry, it suffices to consider the problem for $x\geq 0, |y|<\infty$. As discussed in Refs.[10,11], the solutions of Eqs.(6)–(8) can be assumed as follows:

$$u^{(1)}(x,y)=\frac{2}{\pi}\sum_{i=1}^{3}\lambda_i^{(1)}(-\alpha_1^{(1)}+\alpha_2^{(1)}\lambda_i^{(1)^2})\int_0^\infty A_i(s)s^4\sin(sx)e^{-\lambda_i^{(1)}sy}ds, \tag{12}$$

$$v^{(1)}(x,y)=-\frac{2}{\pi}\sum_{i=1}^{3}(c_{11}^{(1)}\varepsilon_{11}^{(1)}-\alpha_3^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})\int_0^\infty A_i(s)s^4\cos(sx)e^{-\lambda_i^{(1)}sy}ds, \tag{13}$$

$$\phi^{(1)}(x,y)=-\frac{2}{\pi}\sum_{i=1}^{3}(c_{11}^{(1)}e_{15}^{(1)}-\alpha_4^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})\int_0^\infty A_i(s)s^4\cos(sx)e^{-\lambda_i^{(1)}sy}ds, \tag{14}$$

$$u^{(2)}(x,y)=\frac{2}{\pi}\sum_{i=1}^{3}\lambda_i^{(2)}(\alpha_1^{(2)}-\alpha_2^{(2)}\lambda_i^{(2)^2})\int_0^\infty B_i(s)s^4\sin(sx)e^{\lambda_i^{(2)}sy}ds, \tag{15}$$

$$v^{(2)}(x,y)=-\frac{2}{\pi}\sum_{i=1}^{3}(c_{11}^{(2)}\varepsilon_{11}^{(2)}-\alpha_3^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})\int_0^\infty B_i(s)s^4\cos(sx)e^{\lambda_i^{(2)}sy}ds, \tag{16}$$

$$\phi^{(2)}(x,y)=-\frac{2}{\pi}\sum_{i=1}^{3}(c_{11}^{(2)}e_{15}^{(2)}-\alpha_4^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})\int_0^\infty B_i(s)s^4\cos(sx)e^{\lambda_i^{(1)}sy}ds, \tag{17}$$

where $A_i(s)(i=1,2,3)$ and $B_i(s)(i=1,2,3)$ are unknown functions for the upper half plane and the lower half plane, respectively. $\lambda_i^{(j)}(i=1,2,3;j=1,2)$ is the root of the algebraic equation as follows: (Here, it is assumed that the characteristic roots $\lambda_1^{(j)^2}\neq\lambda_2^{(j)^2}\neq\lambda_3^{(j)^2}>0$. The other cases can be obtained using a similar method. They are omitted in the present paper.)

$$a^{(j)}\lambda^{(j)^6}-b^{(j)}\lambda^{(j)^4}+c^{(j)}\lambda^{(j)^2}-d^{(j)}=0, \tag{18}$$

where $a^{(j)},b^{(j)},c^{(j)}$ and $d^{(j)}$ are constants which are dependent on the material properties of the upper half plane and the lower half plane. They can be seen in Refs.[11,12]. They are omitted in the present paper.

Substituting Eqs.(12)–(17) into Eqs.(1)–(5), it can be obtained

$$\begin{aligned}\sigma_y^{(1)}(x,y)=&\frac{2}{\pi}\sum_{i=1}^{3}\lambda_i^{(1)}[c_{13}^{(1)}(-\alpha_1^{(1)}+\alpha_2^{(1)}\lambda_i^{(1)^2})+c_{33}^{(1)}(c_{11}^{(1)}\varepsilon_{11}^{(1)}-\alpha_3^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})\\&+e_{33}^{(1)}(c_{11}^{(1)}e_{15}^{(1)}-\alpha_4^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})]\int_0^\infty A_i(s)s^5e^{-\lambda_i^{(1)}sy}\cos(sx)ds,\end{aligned} \tag{19}$$

$$\begin{aligned}\sigma_{xy}^{(1)}(x,y)=&\frac{2}{\pi}\sum_{i=1}^{3}[c_{44}^{(1)}\lambda_i^{(1)^2}(\alpha_1^{(1)}-\alpha_2^{(1)}\lambda_i^{(1)^2})+c_{44}^{(1)}(c_{11}^{(1)}\varepsilon_{11}^{(1)}-\alpha_3^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})\\&+e_{15}^{(1)}(c_{11}^{(1)}e_{15}^{(1)}-\alpha_4^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})]\int_0^\infty A_i(s)s^5e^{-\lambda_i^{(1)}sy}\sin(sx)ds,\end{aligned} \tag{20}$$

$$\begin{aligned}D_y^{(1)}(x,y)=&\frac{2}{\pi}\sum_{i=1}^{3}\lambda_i^{(1)}[e_{31}^{(1)}(-\alpha_1^{(1)}+\alpha_2^{(1)}\lambda_i^{(1)^2})+e_{33}^{(1)}(c_{11}^{(1)}\varepsilon_{11}^{(1)}-\alpha_3^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})\\&-\varepsilon_{33}^{(1)}(c_{11}^{(1)}e_{15}^{(1)}-\alpha_4^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})]\int_0^\infty A_i(s)s^5e^{-\lambda_i^{(1)}sy}\cos(sx)ds,\end{aligned} \tag{21}$$

$$\begin{aligned}\sigma_y^{(2)}(x,y)=&\frac{2}{\pi}\sum_{i=1}^{3}\lambda_i^{(2)}[c_{13}^{(2)}(\alpha_1^{(2)}-\alpha_2^{(2)}\lambda_i^{(2)^2})-c_{33}^{(2)}(c_{11}^{(2)}\varepsilon_{11}^{(2)}-\alpha_3^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})\\&-e_{33}^{(2)}(c_{11}^{(2)}e_{15}^{(2)}-\alpha_4^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})]\int_0^\infty B_i(s)s^5e^{\lambda_i^{(2)}sy}\cos(sx)ds,\end{aligned} \tag{22}$$

$$\begin{aligned}\sigma_{xy}^{(2)}(x,y)=&\frac{2}{\pi}\sum_{i=1}^{3}[c_{44}^{(2)}\lambda_i^{(2)^2}(\alpha_1^{(2)}-\alpha_2^{(2)}\lambda_i^{(2)^2})+c_{44}^{(2)}(c_{11}^{(2)}\varepsilon_{11}^{(2)}-\alpha_3^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})\\&+e_{15}^{(2)}(c_{11}^{(2)}e_{15}^{(2)}-\alpha_4^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})]\int_0^\infty B_i(s)s^5e^{\lambda_i^{(2)}sy}\sin(sx)ds,\end{aligned} \tag{23}$$

$$\begin{aligned}D_y^{(2)}(x,y)=&\frac{2}{\pi}\sum_{i=1}^{3}\lambda_i^{(2)}[e_{31}^{(2)}(\alpha_1^{(2)}-\alpha_2^{(2)}\lambda_i^{(2)^2})-e_{33}^{(2)}(c_{11}^{(2)}\varepsilon_{11}^{(2)}-\alpha_3^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})\\&+\varepsilon_{33}^{(2)}(c_{11}^{(2)}e_{15}^{(2)}-\alpha_4^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})]\int_0^\infty B_i(s)s^5e^{\lambda_i^{(2)}sy}\cos(sx)ds.\end{aligned} \tag{24}$$

From Eqs.(9)–(11), we see that $\sigma_y^{(1)}(x,0)=\sigma_y^{(2)}(x,0)$, $\sigma_{xy}^{(1)}(x,0)=\sigma_{xy}^{(2)}(x,0)$ and $D_y^{(1)}(x,0)=D_y^{(2)}(x,0)$ for all values of x and it is easily shown that this condition is equivalent to equations:

$$\begin{aligned}&\sum_{i=1}^{3}\lambda_i^{(1)}[c_{13}^{(1)}(-\alpha_1^{(1)}+\alpha_2^{(1)}\lambda_i^{(1)^2})+c_{33}^{(1)}(c_{11}^{(1)}\varepsilon_{11}^{(1)}-\alpha_3^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})\\&+e_{33}^{(1)}(c_{11}^{(1)}e_{15}^{(1)}-\alpha_4^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})]A_i(s)\end{aligned}$$

$$
\begin{aligned}
&=\sum_{i=1}^{3}\lambda_i^{(2)}[c_{13}^{(2)}(\alpha_1^{(2)}-\alpha_2^{(2)}\lambda_i^{(2)^2})-c_{33}^{(2)}(c_{11}^{(2)}\varepsilon_{11}^{(2)}-\alpha_3^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})\\
&\quad-e_{33}^{(2)}(c_{11}^{(2)}e_{15}^{(2)}-\alpha_4^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})]B_i(s),
\end{aligned}\tag{25}
$$

$$
\begin{aligned}
&\sum_{i=1}^{3}[c_{44}^{(1)}\lambda_i^{(1)^2}(\alpha_1^{(1)}-\alpha_2^{(1)}\lambda_i^{(1)^2})+c_{44}^{(1)}(c_{11}^{(1)}\varepsilon_{11}^{(1)}-\alpha_3^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})\\
&\quad+e_{15}^{(1)}(c_{11}^{(1)}e_{15}^{(1)}-\alpha_4^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})]A_i(s)\\
&=\sum_{i=1}^{3}[c_{44}^{(2)}\lambda_i^{(2)^2}(\alpha_1^{(2)}-\alpha_2^{(2)}\lambda_i^{(2)^2})+c_{44}^{(2)}(c_{11}^{(2)}\varepsilon_{11}^{(2)}-\alpha_3^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})\\
&\quad+e_{15}^{(2)}(c_{11}^{(2)}e_{15}^{(2)}-\alpha_4^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})]B_i(s),
\end{aligned}\tag{26}
$$

$$
\begin{aligned}
&\sum_{i=1}^{3}\lambda_i^{(1)}[e_{31}^{(1)}(-\alpha_1^{(1)}+\alpha_2^{(1)}\lambda_i^{(1)^2})+e_{33}^{(1)}(c_{11}^{(1)}\varepsilon_{11}^{(1)}-\alpha_3^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})\\
&\quad-\varepsilon_{33}^{(1)}(c_{11}^{(1)}e_{15}^{(1)}-\alpha_4^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})]A_i(s)\\
&=\sum_{i=1}^{3}\lambda_i^{(2)}[e_{31}^{(2)}(\alpha_1^{(2)}-\alpha_2^{(2)}\lambda_i^{(2)^2})-e_{33}^{(2)}(c_{11}^{(2)}\varepsilon_{11}^{(2)}-\alpha_3^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})\\
&\quad+\varepsilon_{33}^{(2)}(c_{11}^{(2)}e_{15}^{(2)}-\alpha_4^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})]B_i(s).
\end{aligned}\tag{27}
$$

To solve the problem, the jumps of the displacements across the crack surfaces are defined as follows:

$$f_1(x)=u^{(1)}(x,0)-u^{(2)}(x,0),\tag{28}$$

$$f_2(x)=v^{(1)}(x,0)-v^{(2)}(x,0),\tag{29}$$

where $f_1(x)$ is an odd function, $f_2(x)$ is an even function. $f_i(x)(i=1,2)$ is an unknown function of x to be determined by the boundary conditions. However, in the previous works, the unknown function was $\frac{\partial f_i(x)}{\partial x}(i=1,2)$, *i.e.*, the dislocation density function.

Substituting Eqs.(12)–(17) into Eqs.(28) and (29), and applying the Fourier transform and the boundary conditions (9)–(11), it can be obtained:

$$\bar{f}_1(s)/s^4=\sum_{i=1}^{3}\lambda_i^{(1)}(-\alpha_1^{(1)}+\alpha_2^{(1)}\lambda_i^{(1)^2})A_i(s)-\sum_{i=1}^{3}\lambda_i^{(2)}(\alpha_1^{(2)}-\alpha_2^{(2)}\lambda_i^{(2)^2})B_i(s),\tag{30}$$

$$
\begin{aligned}
\bar{f}_2(s)/s^4=&-\sum_{i=1}^{3}(c_{11}^{(1)}\varepsilon_{11}^{(1)}-\alpha_3^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})A_i(s)\\
&+\sum_{i=1}^{3}(c_{11}^{(2)}\varepsilon_{11}^{(2)}-\alpha_3^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})B_i(s),
\end{aligned}\tag{31}
$$

$$-\sum_{i=1}^{3}(c_{11}^{(1)}e_{15}^{(1)}-\alpha_4^{(1)}\lambda_i^{(1)^2}+c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})A_i(s)+\sum_{i=1}^{3}(c_{11}^{(2)}e_{15}^{(2)}-\alpha_4^{(2)}\lambda_i^{(2)^2}+c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})B_i(s)=0.\tag{32}$$

Here a superposed bar indicates the Fourier transform.

By solving Eqs.(25)–(27) and (30)–(32) with six unknown functions and substituting the solutions into Eqs.(19) and (20) and applying the boundary conditions (9) and (10), it can be obtained:

$$\sigma_y^{(1)}(x,0) = \frac{2}{\pi}\int_0^\infty s[\beta_1\bar{f}_1(s) + \beta_2\bar{f}_2(s)]\cos(sx)ds$$
$$= -\sigma_0, \qquad 0 \le x \le l, \tag{33}$$

$$\sigma_{xy}^{(1)}(x,0) = \frac{2}{\pi}\int_0^\infty s[\beta_3\bar{f}_1(s) + \beta_4\bar{f}_2(s)]\sin(sx)ds$$
$$= 0, \qquad 0 \le x \le l, \tag{34}$$

$$\int_0^\infty \bar{f}_1(s)\sin(sx)ds = 0, \qquad x > l, \tag{35}$$

$$\int_0^\infty \bar{f}_2(s)\cos(sx)ds = 0, \qquad x > l, \tag{36}$$

where $\beta_j(j = 1, 2, 3, 4)$ are non-zero constants which are dependent on the material properties of the upper half plane and the lower half plane. It can be seen in Appendix A. When the upper half plane material is the same as the lower half plane material, it can be obtained that $\beta_1 = \beta_4 = 0$ and the solution of the present paper can be returned to the one of the ordinary crack in homogeneous piezoelectric materials. To determine the unknown functions $\bar{f}_1(s)$ and $\bar{f}_2(s)$, the above two pairs of the dual integral equations (33)–(36) must be solved.

3 Solution of Dual Integral Equations

As assumption mentioned above, *i.e.*, it is assumed that the effect of the crack surface overlapping very near the crack tips is negligible. The jumps of the displacements across the crack surface can be expanded by the following series:

$$f_1(x) = \sum_{n=0}^\infty a_n \mathrm{P}_{2n+1}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{\frac{1}{2}}, \qquad \text{for } 0 \le x \le l, \tag{37}$$

$$f_1(x) = 0, \qquad \text{for } l < x, \tag{38}$$

$$f_2(x) = \sum_{n=0}^\infty b_n \mathrm{P}_{2n}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{\frac{1}{2}}, \qquad \text{for } 0 \le x \le l, \tag{39}$$

$$f_2(x) = 0, \qquad \text{for } l < x, \tag{40}$$

where a_n and b_n are unknown coefficients, $\mathrm{P}_n^{(1/2,1/2)}(x)$ is a Jacobi polynomial[13]. The Fourier transform of Eqs.(37)–(40) is[14]

$$\begin{cases} \bar{f}_1(s) = \displaystyle\sum_{n=0}^\infty a_n Q_n \frac{1}{s}\mathrm{J}_{2n+2}(sl), \\ Q_n = \sqrt{\pi}(-1)^n \dfrac{\Gamma(2n+2+\frac{1}{2})}{(2n+1)!}, \end{cases} \tag{41}$$

$$\begin{cases} \bar{f}_2(s) = \displaystyle\sum_{n=0}^\infty b_n R_n \frac{1}{s}\mathrm{J}_{2n+1}(sl), \\ R_n = \sqrt{\pi}(-1)^n \dfrac{\Gamma(2n+1+\frac{1}{2})}{(2n)!}, \end{cases} \tag{42}$$

where $\Gamma(x)$ and $\mathrm{J}_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Eqs.(41) and (42) into Eqs.(33)–(36), it can be shown that Eqs.(35) and (36) are automatically satisfied. After integration with respect to x in $[0, x]$, Eqs.(33) and (34) reduce to

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{1}{s}[\beta_1 a_n Q_n \mathrm{J}_{2n+2}(sl)+\beta_2 b_n R_n \mathrm{J}_{2n+1}(sl)]\sin(sx)ds=-\sigma_0 x, \quad 0\le x\le l, \tag{43}$$

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{1}{s}[\beta_3 a_n Q_n \mathrm{J}_{2n+2}(sl)+\beta_4 b_n R_n \mathrm{J}_{2n+1}(sl)][\cos(sx)-1]ds=0, \quad 0\le x\le l. \tag{44}$$

The semi-infinite integral in Eqs.(43) and (44) can be evaluated directly. Equations.(43) and (44) can now be solved for the coefficients a_n and b_n by the Schmidt method[8,12]. The method is omitted in the present work. It can be seen in Ref.[12]. When the upper half plane material is the same as the lower half plane material, it can be obtained that $\beta_1=\beta_4=0$. Therefore from Eqs.(43) and (44), it can be derived that $a_n=0(n=0,1,2,\cdots)$, $b_0=-\frac{\sigma_0\sqrt{\pi}l}{\beta_2}$ and $b_n=0(n=1,2,3,\cdots)$.

4 Intensity Factors

The coefficients a_n and b_n are known, so that the entire stress field can be obtained. However, in fracture mechanics, it is important to determine the stresses $\sigma_y^{(1)}, \sigma_{xy}^{(1)}$ and the electric displacement $D_y^{(1)}$ in the vicinity of the crack tips. In the case of the present study, $\sigma_y^{(1)}, \sigma_{xy}^{(1)}$ and $D_y^{(1)}$ along the crack line can be expressed as

$$\begin{aligned}\sigma_y^{(1)}(x,0)=&\frac{2}{\pi}\Big\{\sum_{n=0}^{\infty}\beta_1 a_n Q_n\int_0^{\infty}\mathrm{J}_{2n+2}(sl)\cos(sx)ds\\&+\sum_{n=0}^{\infty}\beta_2 b_n R_n\int_0^{\infty}\mathrm{J}_{2n+1}(sl)\cos(sx)ds\Big\},\end{aligned} \tag{45}$$

$$\begin{aligned}\sigma_{xy}^{(1)}(x,0)=&\frac{2}{\pi}\Big\{\sum_{n=0}^{\infty}\beta_3 a_n Q_n\int_0^{\infty}\mathrm{J}_{2n+2}(sl)\sin(sx)ds\\&+\sum_{n=0}^{\infty}\beta_4 b_n R_n\int_0^{\infty}\mathrm{J}_{2n+1}(sl)\sin(sx)ds\Big\},\end{aligned} \tag{46}$$

$$\begin{aligned}D_y^{(1)}(x,0)=&\frac{2}{\pi}\Big\{\sum_{n=0}^{\infty}\beta_5 a_n Q_n\int_0^{\infty}\mathrm{J}_{2n+2}(sl)\cos(sx)ds\\&+\sum_{n=0}^{\infty}\beta_6 b_n R_n\int_0^{\infty}\mathrm{J}_{2n+1}(sl)\cos(sx)ds\Big\},\end{aligned} \tag{47}$$

where β_5 and β_6 are non-zero constants. It can be seen in Appendix A.

When the upper half plane material is the same as the lower half plane material, it can be obtained that $\beta_5=0$ and

$$\begin{aligned}\sigma_{yy}^{(1)}(x,0)&=\frac{2}{\pi}\int_0^{\infty}\beta_2 b_0 R_0 \mathrm{J}_1(sl)\cos(sx)ds\\&=-\sigma_0 l\int_0^{\infty}\mathrm{J}_1(sl)\cos(sx)ds\\&=\begin{cases}-\sigma_0, & x<l,\\ \dfrac{\sigma_0 l^2}{\sqrt{x^2-l^2}[x+\sqrt{x^2-l^2}]}, & x>l,\end{cases}\end{aligned} \tag{48}$$

$$\sigma_{xy}^{(1)}(x,0)=0, \tag{49}$$

$$\begin{aligned} D_y^{(1)}(x,0) &= \frac{2}{\pi}\beta_6 b_0 R_0 \int_0^\infty \mathrm{J}_{2n+1}(sl)\cos(sx)ds \\ &= -\frac{\sigma_0\beta_6 l}{\beta_2}\int_0^\infty \mathrm{J}_1(sl)\cos(sx)ds \\ &= \begin{cases} -\dfrac{\sigma_0\beta_6}{\beta_2}, & x<l, \\ \dfrac{\sigma_0\beta_6 l^2}{\beta_2\sqrt{x^2-l^2}[x+\sqrt{x^2-l^2}]}, & x>l, \end{cases} \end{aligned} \tag{50}$$

for this case, the exact solution of the ordinary crack in the homogeneous piezoelectric materials[11] can be obtained by use of the Schmidt method.

From Eqs.(45)–(47), the singular part of the stress and the electric displacement fields can be expressed respectively as follows ($l<x$):

$$\begin{cases} \sigma = -\dfrac{2\beta_2}{\pi}\displaystyle\sum_{n=0}^\infty b_n R_n H_n(x), \\ \tau = -\dfrac{2\beta_3}{\pi}\displaystyle\sum_{n=0}^\infty a_n Q_n Z_n(x), \\ D = -\dfrac{2\beta_6}{\pi}\displaystyle\sum_{n=0}^\infty b_n R_n H_n(x), \end{cases} \tag{51}$$

where

$$H_n(x)=\frac{(-1)^n l^{2n+1}}{\sqrt{x^2-l^2}[x+\sqrt{x^2-l^2}]^{2n+1}}, \quad Z_n(x)=\frac{(-1)^n l^{2n+2}}{\sqrt{x^2-l^2}[x+\sqrt{x^2-l^2}]^{2n+2}}.$$

The stress intensity factors $K_{\mathrm{I}}, K_{\mathrm{II}}$ and the electric displacement intensity factor K_{D} can be written as follows:

$$K_{\mathrm{I}}=\lim_{x\to l^+}\sqrt{2\pi(x-l)}\cdot\sigma=-\frac{2\beta_2}{\sqrt{l}}\sum_{n=0}^\infty b_n\frac{\Gamma\left(2n+1+\frac{1}{2}\right)}{(2n)!}, \tag{52}$$

$$K_{\mathrm{II}}=\lim_{x\to l^+}\sqrt{2\pi(x-l)}\cdot\tau=-\frac{2\beta_3}{\sqrt{l}}\sum_{n=0}^\infty a_n\frac{\Gamma\left(2n+2+\frac{1}{2}\right)}{(2n+1)!}, \tag{53}$$

$$K_{\mathrm{D}}=\lim_{x\to l^+}\sqrt{2\pi(x-l)}D=-\frac{2\beta_6}{\sqrt{l}}\sum_{n=0}^\infty b_n\frac{\Gamma(2n+1+\frac{1}{2})}{(2n)!}. \tag{54}$$

5 Numerical Calculations and Discussion

As discussed in the works[12,15,16], it can be seen that the Schmidt method is performed satisfactorily if the first ten terms of infinite series in Eqs.(43) and (44) are retained. In all computation, we must firstly examine the relations for the roots $\lambda_j^{(i)}(i=1,2;j=1,2,3)$. Then the constants $\beta_i(i=1,2,3,4,5,6)$ can be obtained. The materials are assumed to be the commercially available PZT-4, P-7 and PZT-19. Material properties are can be obtained in Refs.[7] and [10]. The dimensionless stress and the electric displacement intensity factors are calculated numerically. The results of the present paper are shown in Figs.2–7. From the results, the following observations are very significant.

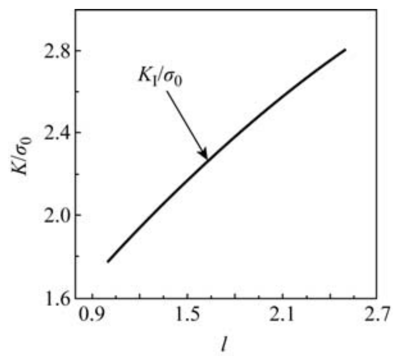

Fig.2 The stress intensity factor *versus* l (Materials of the upper and the lower half planes are both of PZT-4)

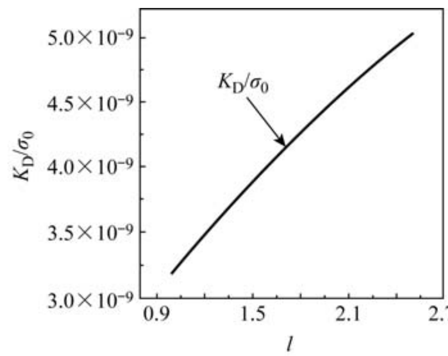

Fig.3 The electric displacement intensity factor *versus* l (Materials of the upper and the lower half planes are both of PZT-4)

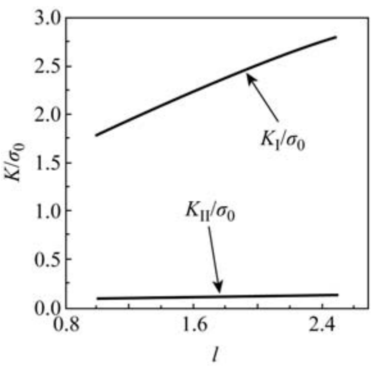

Fig.4 The stress intensity factor *versus* l (Material of the upper half plane is PZT-4 and material of the lower half plane is P-7)

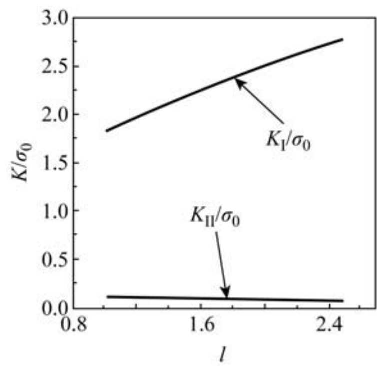

Fig.5 The stress intensity factor *versus* l (Material of the upper half plane is P-7 and material of the lower half plane is PZT-4)

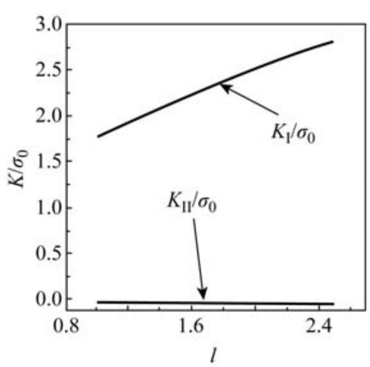

Fig.6 The stress intensity factor *versus* l (Material of the upper half plane is PZT-19 and material of the lower half plane is P-7)

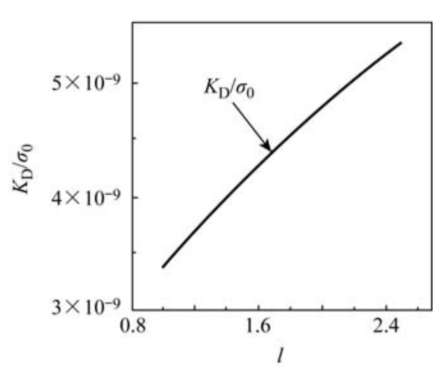

Fig.7 The electric displacement intensity factor *versus* l (Material of the upper half plane is PZT-4 and material of the lower half plane is P-7)

(i) The aim of the present paper is to give an approach to solve the problem of a Mode-I interface crack in piezoelectric materials under the assumptions that the effect of the crack surface overlapping very near the crack tips is negligible. The solving process is quite different from the other works such as in Refs.[1–7] and [10,11]. Contrary to the previous solutions[1,2,6] of the interface crack problems, it is found that the stress singularity of the present interface crack solution is of the same nature as that for the ordinary crack in homogeneous materials. The solution of the present paper can be returned to the one of the ordinary crack in the homogeneous piezoelectric materials[11]. When the upper half plane material is the same as the lower half plane material, the exact solution can be obtained as in Eqs.(48)–(50) and the results given in Figs.2 and 3. In the present paper, the unknown variables of dual integral equations are the displacements across the crack surfaces. However, in the previous works, the unknown variables of dual integral equations are the dislocation density functions. This is the major difference.

(ii) During the solving process, the mathematical difficulties do not meet, *i.e.*, the oscillatory stress singularity and the overlapping of the crack surfaces do not meet.

(iii) The stress intensity factors K_{I}/σ_0 almost linearly increase with increase of the crack length as shown in Fig.4 to Fig.6. However, the shear stress intensity factor K_{II}/σ_0 is very smaller than the stress intensity factor K_{I}/σ_0. The shear stress intensity factor K_{II}/σ_0 may be negative.

(iv) The symbol of the shear stress intensity factors K_{II}/σ_0 will change when the materials of the upper half plane and the lower half plane exchange as shown in Fig.4 and Fig.5. However, the stress intensity factors K_{I}/σ_0 do not change when the materials of the upper half plane and the lower half plane exchange.

(v) It is found that the electric displacement contains singularities at the crack tips in despite of the electric displacement and the electric potential are assumed to be continuous across the crack surfaces. The electric displacement intensity factors K_{D}/σ_0 almost linearly increase with increase of the crack length as shown in Fig.7. The symbol and the value of the electric displacement intensity factors K_{D}/σ_0 will not change when the materials of the upper half plane and the lower half plane exchange. Contrary to the electric displacement intensity factors of the impermeable crack problem[6], it can be found that the electric displacement intensity factors in the present paper are very small for the permeable crack surface conditions as shown in Fig.7.

References

[1] England A H. A crack between dissimilar media[J]. *Journal of Applied Mechanics,* 1965, **32**(3):400–402.

[2] Rice J R. Elastic fracture mechanics concept for interface cracks[J]. *ASME Journal of Applied Mechanics,* 1988, **55**(1):98–103.

[3] Comninou M. The interface crack[J]. *ASME Journal of Applied Mechanics,* 1977, **44**(4):631–636.

[4] Comninou M. The interface crack in shear field[J]. *ASME Journal of Applied Mechanics,* 1978, **45**(2):287–290.

[5] Comninou M, Schmueser D. The interface crack in combined tension compression and shear field[J]. *ASME Journal of Applied Mechanics,* 1979, **46**(3):345–348.

[6] Suo Z, Kuo C M, Barnett D M, Willis J R. Fracture mechanics for piezoelectric ceramics[J]. *Journal of the Mechanics and Physics of Solids,* 1992, **40**(4):739–765.

[7] Herrmann K P, Loboda V V. Fracture mechanical assessment of electrically permeable interface cracks in piezoelectric bimaterials by consideration of various contact zone models[J]. *Archive of Applied Mechanics,* 2000, **70**(1):127–143.

[8] Morse P M, Feshbach H. Methods of Theoretical Physics[M]. McGraw-Hill, New York, 1958, Vol.1, 926.

[9] Erdogan F, Wu B H. Interface crack problems in layered orthotropic materials[J]. *Journal of the Mechanics and Physics of Solids,* 1993, **41**(5):889–917.

[10] Ding H J, Chen B, Liang J. General solutions for coupled equations for piezoelectric media[J]. *International Journal of Solids and Structures,* 1996, **33**(16):2283–2296.

[11] Yang F Q. Fracture mechanics for a Mode I crack in piezoelectric materials[J]. *International Journal of Solids and Structures,* 2001, **38**(21):3813–3830.

[12] Zhou Zhengong, Sun Jianliang, Wang Biao. Investigation of the behavior of a crack in a piezoelectric material subjected to a uniform tension loading by use of the non-local theory[J]. *International Journal of Engineering Science,* 2004, **42**(19/20):2041–2063.

[13] Gradshteyn I S, Ryzhik I M. Table of Integrals, Series and Products[M]. Academic Press, New York, 1980, 923–967.

[14] Erdelyi A. Tables of Integral Transforms[M]. McGraw-Hill, New York, Vol.1, 1954.

[15] Zhou Zhengong, Wang Biao. Investigation of a Griffith crack subject to uniform tension using the non-local theory by a new method[J]. *Applied Mathematics and Mechanics* (*English Edition*), 1999, **20**(10):1099–1107.

[16] Zhou Zhengong, Wang Biao. Investigation of the dynamic behavior of two collinear anti-plane shear cracks in a piezoelectric layer bonded to two half spaces by a new method[J]. *Applied Mathematics and Mechanics* (*English Edition*), 2003, **24**(1):1–13.

Appendix A

For $\lambda_1^{(j)^2} \neq \lambda_2^{(j)^2} \neq \lambda_3^{(j)^2} > 0, (j = 1, 2)$ case, the non-zero constants $\beta_j (j = 1, 2, 3, 4, 5, 6)$ can be obtained by following formulas:

$$\boldsymbol{M} = [m_{ij}], \quad \boldsymbol{Q} = [q_{ij}], \quad \boldsymbol{N} = [n_{ij}], \quad \boldsymbol{P} = [p_{ij}] \qquad (i, j = 1, 2, 3),$$

$$m_{1i} = \lambda_i^{(1)}[c_{13}^{(1)}(-\alpha_1^{(1)} + \alpha_2^{(1)}\lambda_i^{(1)^2}) + c_{33}^{(1)}(c_{11}^{(1)}\varepsilon_{11}^{(1)} - \alpha_3^{(1)}\lambda_i^{(1)^2} + c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})$$
$$+ e_{33}^{(1)}(c_{11}^{(1)}e_{15}^{(1)} - \alpha_4^{(1)}\lambda_i^{(1)^2} + c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})] \qquad (i = 1, 2, 3),$$

$$m_{2i} = c_{44}^{(1)}\lambda_i^{(1)^2}[(\alpha_1^{(1)} - \alpha_2^{(1)}\lambda_i^{(1)^2}) + c_{44}^{(1)}(c_{11}^{(1)}\varepsilon_{11}^{(1)} - \alpha_3^{(1)}\lambda_i^{(1)^2} + c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})$$
$$+ e_{15}^{(1)}(c_{11}^{(1)}e_{15}^{(1)} - \alpha_4^{(1)}\lambda_i^{(1)^2} + c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})] \qquad (i = 1, 2, 3),$$

$$m_{3i} = \lambda_i^{(1)}[e_{31}^{(1)}(-\alpha_1^{(1)} + \alpha_2^{(1)}\lambda_i^{(1)^2}) + e_{33}^{(1)}(c_{11}^{(1)}\varepsilon_{11}^{(1)} - \alpha_3^{(1)}\lambda_i^{(1)^2} + c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4})$$
$$- \varepsilon_{33}^{(1)}(c_{11}^{(1)}e_{15}^{(1)} - \alpha_4^{(1)}\lambda_i^{(1)^2} + c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4})] \qquad (i = 1, 2, 3),$$

$$q_{1i} = \lambda_i^{(2)}[c_{13}^{(2)}(\alpha_1^{(2)} - \alpha_2^{(2)}\lambda_i^{(2)^2}) - c_{33}^{(2)}(c_{11}^{(2)}\varepsilon_{11}^{(2)} - \alpha_3^{(2)}\lambda_i^{(2)^2} + c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})$$
$$- e_{33}^{(2)}(c_{11}^{(2)}e_{15}^{(2)} - \alpha_4^{(2)}\lambda_i^{(2)^2} + c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})] \qquad (i = 1, 2, 3),$$

$$q_{2i} = c_{44}^{(2)}\lambda_i^{(2)^2}[(\alpha_1^{(2)} - \alpha_2^{(2)}\lambda_i^{(2)^2}) + c_{44}^{(2)}(c_{11}^{(2)}\varepsilon_{11}^{(2)} - \alpha_3^{(2)}\lambda_i^{(2)^2} + c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})$$
$$+ e_{15}^{(2)}(c_{11}^{(2)}e_{15}^{(2)} - \alpha_4^{(2)}\lambda_i^{(2)^2} + c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})] \qquad (i = 1, 2, 3),$$

$$q_{3i} = \lambda_i^{(2)}[e_{31}^{(2)}(\alpha_1^{(2)} - \alpha_2^{(2)}\lambda_i^{(2)^2}) - e_{33}^{(2)}(c_{11}^{(2)}\varepsilon_{11}^{(2)} - \alpha_3^{(2)}\lambda_i^{(2)^2} + c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4})$$
$$+ \varepsilon_{33}^{(2)}(c_{11}^{(2)}e_{15}^{(2)} - \alpha_4^{(2)}\lambda_i^{(2)^2} + c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4})] \qquad (i = 1, 2, 3),$$

$$n_{1i} = \lambda_i^{(1)}(-\alpha_1^{(1)} + \alpha_2^{(1)}\lambda_i^{(1)^2}) \qquad (i = 1, 2, 3),$$

$$n_{2i} = -(c_{11}^{(1)}\varepsilon_{11}^{(1)} - \alpha_3^{(1)}\lambda_i^{(1)^2} + c_{44}^{(1)}\varepsilon_{33}^{(1)}\lambda_i^{(1)^4}) \qquad (i = 1, 2, 3),$$

$$n_{3i} = -(c_{11}^{(1)}e_{15}^{(1)} - \alpha_4^{(1)}\lambda_i^{(1)^2} + c_{44}^{(1)}e_{33}^{(1)}\lambda_i^{(1)^4}) \qquad (i = 1, 2, 3),$$

$$p_{1i} = -\lambda_i^{(2)}(\alpha_1^{(2)} - \alpha_2^{(2)}\lambda_i^{(2)^2}) \qquad (i = 1, 2, 3),$$

$$p_{2i} = (c_{11}^{(2)}\varepsilon_{11}^{(2)} - \alpha_3^{(2)}\lambda_i^{(2)^2} + c_{44}^{(2)}\varepsilon_{33}^{(2)}\lambda_i^{(2)^4}) \qquad (i = 1, 2, 3),$$

$$p_{3i} = c_{11}^{(2)}e_{15}^{(2)} - \alpha_4^{(2)}\lambda_i^{(2)^2} + c_{44}^{(2)}e_{33}^{(2)}\lambda_i^{(2)^4} \qquad (i = 1, 2, 3),$$

$$\begin{bmatrix} \beta_1 & \beta_2 & * \\ \beta_3 & \beta_4 & * \\ \beta_5 & \beta_6 & * \end{bmatrix} = \boldsymbol{M}(\boldsymbol{N} + \boldsymbol{P}\boldsymbol{Q}^{-1}\boldsymbol{M})^{-1}.$$

For the other cases, the constants $\beta_j (j = 1, 2, 3, 4, 5, 6)$ can be obtained by using the same method. They are omitted in the present paper.